THE NOTION OF THE GREEN'S FUNCTION IN THE THEORY OF INTEGRO-DIFFERENTIAL EQUATIONS. II*

BY

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1. Problem and hypotheses. We are concerned here with an integro-differential boundary value problem (★) consisting of an integro-differential equation

$$(1.1)\qquad \begin{aligned} L(u) &\equiv u^{(n)} + P_1(x, \rho)u^{(n-1)} + \cdots + P_n(x, \rho)u \\ &= f(x) + \int_a^b h(x, \xi; \rho)u(\xi)d\xi \end{aligned}$$

together with n linear boundary conditions

$$(1.2)\qquad L_i(u) \equiv A_i(u, \rho) + B_i(u, \rho) + \int_a^b \alpha_i(x; \rho)u(x)dx = 0;$$

$$(1.3)\qquad \begin{aligned} P_i(x, \rho) &\equiv \rho^i p_{i0}(x) + \rho^{i-1}p_{i1}(x) + \cdots + p_{ii}(x), \\ A_i(u, \rho) &\equiv \sum_{s=0}^{n} \rho^s A_i^{(s)}(u), \quad B_i(u, \rho) \equiv \sum_{s=0}^{n} \rho^s B_i^{(s)}(u), \\ \alpha_i(x; \rho) &\equiv \sum_{s=0}^{n} \rho^s \alpha_{is}(x) \end{aligned}$$

[I; 10, 13, 19]. The coefficients $\phi_j(x;\rho)$ of [I; 19] are assumed here to be identically zeros while the function $h(x,\xi;\rho)$ is defined by

$$(1.4)\qquad h(x, \xi; \rho) \equiv p_{n0}(x) \sum_{s=0}^{n} \rho^s h_s(x, \xi).$$

We assume that our problem (★) satisfies hypotheses (**A**) and (**B**) of [I; 10, 13] while hypothesis (**C**) of (I; 19] is replaced by

(**C**) i. The function $h_n(x, \xi)$ has partial derivatives of the first order

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† The present paper is a continuation of a paper published in these Transactions (vol. 29 (1927), pp. 755–800) under the same title. We shall refer by [I] to this paper as to notation and bibliography of the subject. See also Errata to [I] at the end of the present Note.

which are of bounded variation in (x, ξ).* The function $h_{n-1}(x, \xi)$ is of bounded variation in (x, ξ). The remaining functions $h_{n-2}(x, \xi), \cdots, h_0(x, \xi)$ are bounded and integrable on (a, b) with respect either to x, or to ξ, or to (x, ξ).

ii. The function $h_n(x, \xi)$ considered as a kernel of a Fredholm integral equation possesses a reciprocal $\mathfrak{h}_n(x, \xi)$ as defined by

$$h_n(x, \xi) + \mathfrak{h}_n(x, \xi) = (h_n \cdot \mathfrak{h}_n)(x, \xi) = (\mathfrak{h}_n \cdot h_n)(x, \xi). \dagger$$

2. **A lemma of the theory of integral equations.** The following well known lemma‡ will be very useful for our discussion.

LEMMA 1. *If a kernel $K(x, \xi)$ is equal to the sum of two other kernels,*

$$K(x, \xi) = K_1(x, \xi) + K_2(x, \xi) \tag{2.1}$$

and if the reciprocals of the kernel $K_1(x, \xi)$ and of the kernel

$$K_0(x, \xi) \equiv K_2(x, \xi) - (\mathfrak{K}_1 \cdot K_2)(x, \xi) \tag{2.2}$$

exist, then the reciprocal of $K(x, \xi)$ exists and is given by

$$\mathfrak{K}(x, \xi) = \mathfrak{K}_1(x, \xi) + \mathfrak{K}_0(x, \xi) - (\mathfrak{K}_0 \cdot \mathfrak{K}_1)(x, \xi). \S \tag{2.3}$$

Formula (2.3) is easily proved if we write the integral equation

$$y(x) = f(x) + K \cdot y(x)$$

in the form

$$y(x) = f(x) + K_2 \cdot y(x) + K_1 \cdot y(x),$$

whence, by definition of the reciprocal and by the hypothesis of Lemma 1,

$$\begin{aligned} y(x) &= f(x) + K_2 \cdot y(x) - \mathfrak{K}_1 \cdot \{f(x) + K_2 \cdot y(x)\} \\ &= f(x) - \mathfrak{K}_1 \cdot f(x) + K_0 \cdot y(x), \\ y(x) &= f(x) - \mathfrak{K}_1 \cdot f(x) - \mathfrak{K}_0 \cdot \{f(x) - \mathfrak{K}_1 \cdot f(x)\}, \end{aligned}$$

which, being compared with

* A function $f(x, \xi)$ will be said to be of bounded variation in (x, ξ) if it is of bounded variation in ξ for each x, and in x for each ξ, the total variation being in each case uniformly bounded on (a, b). A function will be designated simply as integrable if it possesses the properties of the functions $h_{n-2}, \cdots, h_0$ below.

† The notation $A \cdot B \equiv (A \cdot B)(x, \xi)$, $A \cdot f(x)$ will be used throughout this paper to designate the "composition" of two kernels $\int_a^b A(x, s)B(s, \xi)ds$, or, respectively, the composition of a kernel and a function, $\int_a^b A(x, s)f(s)ds$. The reciprocal of a given kernel will be usually designated by the corresponding letter of the German alphabet.

‡ Hellinger-Toeplitz, *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten*, Encyklopädie der Mathematischen Wissenschaften, II_3, 1927, pp. 1379–1380.

§ It is hardly necessary to specify various conditions of integrability which should be added in the statement of Lemma 1.

$$y(x) = f(x) - \mathfrak{K} \cdot f(x),$$

yields the desired result.

3. **The extension of certain results of [I].** As in [I; 15] we define

$$(3.1)\qquad T_1(f;\, x,\, \xi) \equiv \rho^{n-1} \int_a^b G(x,\, t;\, \rho) p_{n0}(t) f(t,\, \xi) dt,$$

$$(3.2)\qquad T_2(f;\, x,\, \xi) \equiv \rho^{n-1} \int_a^b G(t,\, x;\, \rho) p_{n0}(t) f(\xi,\, t) dt.$$

The symbol $\Lambda(u;\, x,\, \xi)$ will be used as a generic notation to designate a linear form in n arguments $u_1,\, u_2,\, \cdots,\, u_n$,

$$\Lambda(u;\, x,\, \xi) \equiv \sum_{i=1}^{n} u_i \lambda_i(x,\, \xi;\, \rho)$$

whose coefficients $\lambda_i(x,\, \xi;\, \rho)$ are defined as sums of a finite number of products of the form

$$[\phi(x)\psi(\xi)]E(\rho),*$$

the functions $\phi(x)$, $\psi(\xi)$ being continuous and of bounded variation on $(a,\, b)$. Likewise, $\Lambda(u;\, x)$ will designate a linear form

$$\Lambda(u;\, x) \equiv \sum_{i=1}^{n} u_i \lambda_i(x;\, \rho)$$

whose coefficients $\lambda_i(x;\, \rho)$ are sums of a finite number of terms $[\phi(x)]E$, where E may depend on ρ and other variables, including x. With this notation we have [I; 15]

LEMMA 2. (i) *If $f(x,\, \xi)$ is any function integrable on $(a,\, b)$, then the integrals*

$$(3.3)\qquad \rho^{-s} D_x^s T_1(f;\, x,\, \xi) \quad (s = 0,\, 1,\, \cdots,\, n-1);\quad T_2(f;\, x,\, \xi)$$

tend uniformly (in x, ξ and ρ) to zero as $|\rho| \to \infty$ and ρ remains in $(\mathfrak{D}_\delta)$.

(ii) *If $f(x,\, \xi)$ is any function of bounded variation on $(a,\, b)$ then*

$$(3.4)\qquad \begin{aligned} |\rho^{-s} D_x^s T_1(f;\, x,\, \xi)| &< N V_f / |\rho| \qquad (s = 0,\, 1,\, \cdots,\, n-1); \\ |T_2(f;\, x,\, \xi)| &< N V_f / |\rho|, \end{aligned}$$

where the constants N, V_f have the same meaning as in [I; 15].

(iii) *If $f(x,\, \xi)$ is any function whose first derivatives are of bounded variation on $(a,\, b)$, then, in $(\mathfrak{D}_\delta)$,*

* By $E(\rho)$, $E(\rho,\, x)$, $E(\rho,\, x,\, \xi)$, $\cdots$ we designate functions which remain uniformly bounded on the respective ranges of their arguments.

$$(3.5)\qquad \begin{aligned} T_1(f;\,x,\,\xi) &= f(x,\,\xi)/\rho + \Lambda(\tilde{\omega}_x';\,x,\,\xi)/\rho + E/\rho^2,\\ T_2(f;\,x,\,\xi) &= f(\xi,\,x)/\rho + \Lambda(\tilde{\omega}_x'';\,x,\,\xi)/\rho + E/\rho^2, \end{aligned}$$

while

$$(3.6)\qquad \rho^{-s+1}D_x^s T_1(f;\,x,\,\xi) = \Lambda(\tilde{\omega}_x';\,x,\,\xi) + E/\rho^2 \qquad (s = 1,\,2,\,\cdots,\,n-1).$$

The proof of Lemma 2 is obtained in precisely the same manner as the proof of the corresponding Lemma 1 of [I; 15]. The foot-note on page 772 of [I] should be used in deriving (3.6).*

Remark. Lemma 2 can be applied even when $f(x,\,\xi)$ is replaced by a function of the type $\Lambda(u;\,x,\,\xi)$ with the result

$$(3.7)\qquad \begin{aligned} p_{n0}(\xi)\rho^{n-1}\int_a^b \Lambda_0(\tilde{\omega}_x';\,x,\,s)G(s,\,\xi;\,\rho)ds &= \Lambda_0(\tilde{\omega}_x';\,x,\,\xi)/\rho\\ &+ \sum_{i=1}^{n}\tilde{\omega}_{i\,\xi}''E(x;\,\rho)/\rho + \rho^{n-2}(E\cdot G)(x,\,\xi;\,\rho) + E/\rho^2, \end{aligned}$$

the function Λ_0 being the same on both sides of (3.7).

LEMMA 3. *The following rules may be used when operating with the symbol* Λ:

(i) $\Lambda_1(\tilde{\omega};\,x,\,\xi)+\Lambda_2(\tilde{\omega};\,x,\,\xi)=\Lambda(\tilde{\omega};\,x,\,\xi)$.

(ii) *If* $f(x,\,\xi)$ *is any function of bounded variation on* $(a,\,b)$, *then*

$$(3.8)\qquad \int_a^b \Lambda(\tilde{\omega}_s;\,s,\,x)f(s,\,\xi)ds = E/\rho.$$

(iii) *Under the hypothesis of* (ii)

$$(3.9)\qquad \int_a^b \Lambda(\tilde{\omega}_x;\,x,\,s)f(s,\,\xi)ds = \Lambda(\tilde{\omega}_x;\,x,\,\xi),$$

$$(\text{iv})\qquad \int_a^b \Lambda_1(\tilde{\omega}_x;\,x,\,s)\Lambda_2(\tilde{\omega}_s;\,s,\,\xi)ds = E/\rho.$$

The rules (i) and (iii) follow immediately from the definition of the symbol Λ. Rule (ii) follows from Lemma 3 of [I; 16], and rule (iv) is readily proved by using the definition of the symbol Λ and applying rule (ii).

4. The formal expression for the Green's function of problem (★). Let $G(x,\,\xi;\,\rho)$ denote, as usual, the Green's function of the associated differential boundary value problem

$$L(u) = f;\quad L_i(u) = 0 \qquad (i = 1,\,2,\,\cdots,\,n).$$

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* The subscripts x, t will be used to designate the dependence of the quantities $\tilde{\omega}_i'$, $\tilde{\omega}_i''$ of [I; 14] respectively on x or on t. If the distinction between $\tilde{\omega}'$ and $\tilde{\omega}''$ is not essential, we shall use simply $\tilde{\omega}$.

If ρ is not a pole of $G(x, \xi; \rho)$ then our problem $(\star)$ is equivalent to the integral equation

$$u(x) = F(x; \rho) + H \cdot u(x; \rho)$$

where

$$F(x; \rho) \equiv G \cdot f(x), \tag{4.1}$$

$$H(x, \xi; \rho) \equiv (G \cdot h)(x, \xi; \rho). \tag{4.2}$$

If the reciprocal $\mathfrak{H}(x, \xi; \rho)$ of the kernel $H(x, \xi; \rho)$ exists, then

$$u(x) = F(x; \rho) - \mathfrak{H} \cdot F(x, \rho) = (G - \mathfrak{H} \cdot G) \cdot f(x).$$

Hence the Green's function $\Gamma(x, \xi; \rho)$ of the problem $(\star)$ is given by

$$\Gamma(x, \xi; \rho) = (G - \mathfrak{H} \cdot G)(x, \xi; \rho) = G(x, \xi; \rho) - \int_a^b \mathfrak{H}(x, s; \rho) G(s, \xi; \rho) ds. \tag{4.3}$$

Therefore the whole question is reduced to the discussion of the kernels $H(x, \xi; \rho)$ and $\mathfrak{H}(x, \xi; \rho)$. This discussion can be carried through if we restrict ρ to range over the domain $(\mathfrak{T})$ [I; 23], on the basis of the known behavior of $G(x, \xi; \rho)$ on $(\mathfrak{T})$ and of the results of the preceding §§2 and 3. The restriction above concerning ρ will be assumed in the sequel without being mentioned explicitly.

5. The discussion of the equivalent integral equation. We start with the kernel $H(x, \xi; \rho)$. On substituting expression (1.4) for $h(x, \xi; \rho)$ into (4.2) we see at once that

$$H(x, \xi; \rho) = \rho T_1(h_n; x, \xi) + T_1(h_{n-1}; x, \xi) + \sum_{s=2}^{n} \rho^{-s+1} T_1(h_{n-s}; x, \xi).$$

Hence, by Lemma 2,

$$H(x, \xi; \rho) = h_n(x, \xi) + l(\tilde{\omega}_x'; x, \xi) + E/\rho, \tag{5.1}$$

$$T_1(h_n; x, \xi) = h_n(x, \xi)/\rho + l(\tilde{\omega}_x'; x, \xi)/\rho + E/\rho^2, \tag{5.2}$$

where $l(\tilde{\omega}_x'; x, \xi)$ is an expression of type Λ.*

We proceed now to the reciprocal $\mathfrak{H}(x, \xi; \rho)$. We set in Lemma 1

$$K_1(x, \xi) = h_n(x, \xi), \quad K_2(x, \xi) = l(\tilde{\omega}_x'; x_1, \xi) + E/\rho.$$

Since the reciprocal $\mathfrak{K}_1(x, \xi) = \mathfrak{h}_n(x, \xi)$ exists by hypothesis we have only to investigate the existence of the reciprocal of the kernel

$$K_0 \equiv l(\tilde{\omega}_x'; x, \xi) - (\mathfrak{h}_n \cdot l)(x, \xi; \rho) + E/\rho = l(\tilde{\omega}_x'; x, \xi) + E/\rho.$$

* $l(\tilde{\omega}_x'; x, \xi)$ will *not* be used as a generic notation. It is completely determined by formula (5.2).

By Lemma 3 the iterated kernel

$$K_0^{(2)}(x, \xi; \rho) = E/\rho.$$

Since this can be made as small as we please by taking $|\rho|$ sufficiently large, the reciprocal $\mathfrak{K}_0''(x, \xi; \rho)$ of the kernel $K_0^{(2)}(x, \xi;\rho)$ exists in $(\mathfrak{T})$ and is itself of the type E/ρ. But, from

$$u = f + K_0 \cdot f = f + K_0 \cdot f + K_0^2 \cdot u,$$
$$u = f + K_0 \cdot f - \mathfrak{K}_0'' \cdot (f + K_0 \cdot f) = f + (K_0 - \mathfrak{K}_0'' - \mathfrak{K}_0'' \cdot K_0) \cdot f$$

it is readily seen that

$$\mathfrak{K}_0 = - K_0 + \mathfrak{K}_0'' + \mathfrak{K}_0'' \cdot K_0 = - l(\bar{\omega}_x' ; x, \xi) + E/\rho.$$

Substitution into (2.3) gives now

$$(5.3)\quad \mathfrak{H}(x, \xi; \rho) \equiv \mathfrak{K} = \mathfrak{h}_n(x, \xi) - l(\bar{\omega}_x' ; x, \xi) + (l \cdot \mathfrak{h}_n)(x, \xi; \rho) + E/\rho.$$

We can prove now the following

THEOREM 1. *Under the hypotheses* (**A**),(**B**),(**C**) *above, the Green's function* $\Gamma(x, \xi; \rho)$ *of the problem* (★) *is meromorphic in* ρ; *furthermore, in* $(\mathfrak{T})$,

$$(5.4)\qquad \Gamma(x, \xi; \rho) - G(x, \xi; \rho) = - (\mathfrak{H} \cdot G)(x, \xi; \rho) = E/\rho^n.$$

The first statement of Theorem 1 is proved in precisely the same manner as the corresponding statement of Theorem 3 of [I; 21]. The second statement follows immediately from (4.3), (5.3) if we apply Lemmas 2 and 3, and Remark to Lemma 2.

COROLLARY. *Under the hypotheses of Theorem* 1, *Theorems* 5 *and* 7 *of* [I; 23, 26] *also hold.*

6. **The equiconvergence theorem.** It was stated in [I, p. 788]† that in the equiconvergence theorem the Birkhoff integral

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1} d\rho \int_a^b \Gamma(x, t; \rho) p_{n0}(t) f(t) dt$$

has to be replaced in the present theory by the integral

$$(6.1)\qquad I_R^*(f) \equiv \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1} d\rho \left\{ \int_a^b f(t) dt \left(\Gamma(x, t; \rho) p_{n0}(t) - \int_a^b \Gamma(x, \xi; \rho) h_n(\xi, t) p_{n0}(\xi) d\xi \right) \right\}.$$

This leads to the

† See also the paper of Lichtenstein referred to in the foot-note ‡ on page 755 of [I].

THEOREM 2. *Under the hypotheses* (**A**), (**B**), (**C**) *above, if* (C_R) *denotes a circle in* $(\mathfrak{D})$ *about the origin, the difference*

$$(6.2)\qquad I_R(f) \equiv I_R^*(f) - \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b G(x, t; \rho)p_{n0}(t)f(t)dt$$

tends to zero as $R\to\infty$, *for any integrable function* $f(x)$, *and uniformly in* x *on* (a, b).

Set

$$(6.3)\qquad \phi_R(x, t) \equiv \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho\left\{p_{n0}(t)(\Gamma(x, t; \rho) - G(x, t; \rho)) - \int_a^b \Gamma(x, \xi; \rho)h_n(\xi, t)p_{n0}(\xi)d\xi\right\}.$$

Then

$$(6.4)\qquad I_R(f) = \int_a^b \phi_R(x, t(f(t)dt,$$

and, by the classical theorem of Lebesgue,† Theorem 2 will be proved if we prove the following:

(i) The function $\phi_R(x, t)$ is uniformly bounded for (C_R) in $(\mathfrak{D})$, x, t on (a, b).

(ii) As $R\to\infty$ the integral

$$\int_\alpha^\beta \phi_R(x, t)dt$$

tends to zero, uniformly in (x, α, β) on (a, b).

To prove (i) we observe that the integral of the second line of (6.3) reduces to $\rho^{1-n}T_1(h_n; x, t)$ if $\Gamma(x, \xi; \rho)$ be replaced by $G(x, \xi; \rho)$. Since by Lemma 2, $T_1(h_n; x, t) = E/\rho$ and, by Theorem 1, $\Gamma - G = E/\rho^n$, we have

$$\phi_R(x, t) = \int_{(C_R)} (E/\rho)d\rho$$

which proves (i).

To prove (ii) we have, in view of (4.3) and (3.1),

$$(6.5)\qquad \begin{aligned} 2\pi i\phi_R(x, t) = &- p_{n0}(t)\int_{(C_R)} \rho^{n-1}(\mathfrak{H}\cdot G)(x, t; \rho)d\rho - \int_{(C_R)} T_1(h_n; x, t)d\rho \\ &+ \int_{(C_R)} \rho^{n-1}(\mathfrak{H}\cdot G)\cdot h_n'(x, t)d\rho, \end{aligned}$$

† *Sur les intégrales singulières*, Annales de Toulouse, (3), vol. 1 (1909), pp. 25–117 (52, 68).

where

$$h_n'(x, \xi) = p_{n0}(x) h_n(x, \xi).$$

Now, by (5.3),

$$(\mathfrak{H}\cdot G)(x, t; \rho) = (\mathfrak{h}_n \cdot G)(x, t; \rho) - (l \cdot G)(x, t; \rho) + (l \cdot \mathfrak{h}_n \cdot G)(x, t; \rho) + (E\cdot G)(x, t; \rho)/\rho.$$

Since $(l \cdot \mathfrak{h}_n)(x, \xi; \rho)$ is of the type $\Lambda(\tilde{\omega}_x'; x, \xi)$, an easy application of Lemma 2, Remark to Lemma 2, and Lemma 3 gives

$$p_{n0}(t)\rho^n(\mathfrak{H}\cdot G)(x, t; \rho) = \mathfrak{h}_n(x, t) + \Lambda(\tilde{\omega}_t''; x, t) - l(\tilde{\omega}_x'; x, t) + (l\cdot\mathfrak{h}_n)(x, t; \rho) + \sum_{i=1}^{n} \tilde{\omega}''_{i\,t} E(x, \rho) + \rho^{n-1}(E\cdot G)(x, t; \rho) + E/\rho.$$

Again, by (5.2),

$$\rho T_1(h_n; x, t) = h_n(x, t) + l(\tilde{\omega}_x'; x, t) + E/\rho,$$

and, finally,

$$\begin{aligned} \rho^n(\mathfrak{H}\cdot G)\cdot h_n'(x, t) &= \rho\mathfrak{H}\cdot T_1(h_n; x, t) \\ &= (\mathfrak{h}_n - l + l\cdot\mathfrak{h}_n + E/\rho)\cdot(h_n + l + E/\rho)(x, t) \\ &= (\mathfrak{h}_n\cdot h_n)(x, t) - (l\cdot h_n)(x, t; \rho) + (l\cdot\mathfrak{h}_n\cdot h_n)(x, t; \rho) + E/\rho. \end{aligned}$$

It should be observed that in these formulas we have to consider only the leading terms in the brackets of the coefficients of the linear forms l and Λ, the contributions of the remaining parts being included in the expression E/ρ.

On collecting all these results and taking in account the reciprocity relations

$$h_n + \mathfrak{h}_n = \mathfrak{h}_n \cdot h_n = h_n \cdot \mathfrak{h}_n$$

we see at once

$$2\pi i\phi_R(x, t) = \int_{(C_R)} \left\{ \Lambda(\tilde{\omega}_t''; x, t) + \sum_{j=1}^{n} \tilde{\omega}_{jt}'' \, E(x, \rho) + \rho^{n-1}(E\cdot G)(x, t; \rho) + \frac{E}{\rho} \right\} \frac{d\rho}{\rho}.$$

It is readily seen however that

$$\int_\alpha^\beta \Lambda(\tilde{\omega}_t''; x, t) dt = \frac{E}{\rho}; \quad \int_\alpha^\beta \tilde{\omega}_{jt}'' \, dt = \frac{E}{\rho};$$

$$\int_\alpha^\beta \rho^{n-1}(E\cdot G)(x, t; \rho) dt = \frac{E}{\rho}.$$

Hence

$$\int_{\alpha}^{\beta} \phi_R(x, t)dt = \int_{(C_R)} E\rho^{-2}d\rho,$$

which yields the desired result.

The reader will find no difficulty now in extending Theorem 6 of [I; 24] as well as in applying the above results to various problems of expansion of an arbitrary function in a series of fundamental functions of the problem (★), and of the summability of such series. Our results are also readily extended to the case where the right-hand member of the integro-differential equation (1.1) has the more general form considered in [I; 19–26]. This extension (in the case where instead of a single integro-differential equation of nth order we deal with a system of n integro-differential equations of the first order) is treated in a Thesis by F. C. Jonah, which will be published elsewhere.

Errata corrigenda to the paper [I]

Page 769, foot-note: The integrand should be replaced by

$$\{\mathfrak{F}_1(\theta_i) - \tfrac{1}{2}(\partial\mathfrak{F}'(\theta_i)/\partial x)\}/\mathfrak{F}'(\theta_i).$$

Pages 763–764: The expressions for the pseudo-resolvent used in the text are correct only in the case of real kernels. Since it is essential here to consider complex-valued kernels as well, the following modifications should be made: ω_k', ω_k of the last formula on page 763, and in the formulas on lines 3, 11, 16, page 764, should be replaced by their conjugates, $\bar{\omega}_k'$, $\bar{\omega}_k$. The same remark concerns ω_s on 8th line.

Pages 774–775: The right-hand member of (29) should be replaced by

$$\sum_j [\omega_{ik0j}(x, t)]E_{ij}(\rho),$$

and ω_{ik0}, E_{ik} in the next two lines should be replaced by ω_{ik0j}, E_{ij} respectively. An analogous correction should be made in the formula (34). In (30) $\tilde{\omega}_i$ should be replaced by $\tilde{\omega}_i'$.

Pages 785–786: It is not always possible to draw the family of contours (C_R) in $(\mathfrak{D})$ in such a way that one and only one pole of the Green's function be enclosed between any two consecutive contours. Hence the terms in the series expansion of the Corollary of 25 should be grouped, each group containing the terms that correspond to sets of poles between the consecutive contours (C_R). This situation, which was stated correctly in the author's paper referred to in the foot-note § on page 755, is overlooked in [I] and in the author's paper referred to by "D" in [I]. The same remark, apparently, should be made concerning all the publications on the subject by various other authors.

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